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A new look at the ω-equation

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SUMMARY

In the conventional quasi-geostrophic form of the 'omega'-equation, the forcing of vertical velocity is usually expressed as the sum of two terms associated respectively with vorticity and temperature advection. Consideration of each term in isolation is misleading and there can be a large degree of cancellation. On the other hand, in Sutcliffe's development theory, this forcing is, in effect, represented by a single term. However, this is achieved at the expense of neglecting another term which is dominant in frontal regions. An investigation, based upon the governing equations, of the manner in which geostrophic balance tends to destroy itself, reveals a simple, concise, one-term representation of the geostrophic forcing of ageostrophic motion. Many of the traditional synoptic rules are then simple deductions from this theory. An application of the theory in the case of a rapidly developing system is demonstrated using a 700 mb chart.

1. INTRODUCTION

In 1947 Sutcliffe produced his theory of development. He derived, subject to a series of approximations, an expression for the difference between the horizontal divergence at two levels, and obtained criteria for upward motion. Much of the present-day reasoning employed by synoptic meteorologists and forecasters follows from Sutcliffe's work. About the same time, Charney (1947) and Eady (1949) were producing their theories of baroclinic instability. The quasi-geostrophic theory that they used has been, perhaps, the cornerstone of modern dynamical meteorology. Implied in this theory is an equation for the vertical velocity – the ω-equation.

We thus have synoptic and dynamic versions of a vertical velocity equation applicable to large- and medium-scale mid-latitude flow. The purpose of this paper is to show how the ω-equation can be written in a simpler form that may easily be interpreted on the synoptician's chart, and also to point out the connection with the more approximated development theory of Sutcliffe.

2. THE CONVENTIONAL QUASI-GEOSTROPHIC ω-EQUATION AND ITS DRAWBACKS

It is convenient to use a pressure-type vertical coordinate $z, = (R\theta_0/g\kappa)(1-(p/p_0)^\kappa)$, which is little different from physical height in the lower troposphere (actually the same as physical height in an atmosphere with a dry adiabatic lapse rate) and simplifies the thermal wind relation. θ_0 and p_0 are standard values of potential temperature and pressure. R is the gas constant, and $\kappa = R/c_p$. The Boussinesq adiabatic equations may be written (see Hoskins and Bretherton 1972)

$$Du/Dt - fv + \partial\phi/\partial x = 0$$

$$Dv/Dt + fu + \partial\phi/\partial y = 0$$

$$D\theta/Dt = 0$$

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

$$\partial\phi/\partial z = (g/\theta_0)\theta.$$

Here, ϕ is the geopotential, and for the present we take the Coriolis parameter, f, to be constant. The synoptician, accustomed to working with height fields at given pressure levels, may note the close correspondence of this set of equations with the more usual pressure coordinate formulation of the primitive equations.

The geostrophic velocities are $u_g = -(1/f)(\partial\phi/\partial y)$ and $v_g = (1/f)(\partial\phi/\partial x)$; and the thermal wind relations $-f(\partial u_g/\partial z) = (g/\theta_0)(\partial\theta/\partial y)$ and $f(\partial v_g/\partial z) = (g/\theta_0)(\partial\theta/\partial x)$. Defining the geostrophic vertical component of relative vorticity, ξ_g, as $\partial v_g/\partial x - \partial u_g/\partial y$, we have

$$f(\partial\xi_g/\partial z) = (g/\theta_0)\nabla_h^2\theta \qquad (1)$$

The forms of the vorticity and temperature equations at the level of quasi-geostrophic theory are $(\partial/\partial t + \mathbf{V}_g.\nabla)\xi_g = f\partial w/\partial z$, and $(\partial/\partial t + \mathbf{V}_g.\nabla)\theta = -w\,d\Theta/dz$, where $\Theta(z)$ is a standard potential temperature distribution with squared buoyancy frequency, N^2, $= (g/\theta_0)(d\Theta/dz)$. Eliminating the time derivative using Eq. (1) gives

$$N^2\nabla_h^2 w + f^2\frac{\partial^2 w}{\partial z^2} = \underset{\mathscr{A}}{f\frac{\partial}{\partial z}(\mathbf{V}_g.\nabla\xi_g)} - \underset{\mathscr{B}}{\frac{g}{\theta_0}\nabla_h^2(\mathbf{V}_g.\nabla\theta)} \qquad (2)$$

For our system of equations this is the form taken by the usual ω-equation. The forcing of vertical motion is by the vertical derivative of vorticity advection (term $\mathscr{A}$) and the horizontal Laplacian of thermal advection (term $\mathscr{B}$).

The drawback of this form for the forcing is that there can be large cancellation between the two terms. Mathematically, the term $f\mathbf{V}_g.\nabla(\partial\xi_g/\partial z)$ can be seen to cancel between them. Adding a speed U on to the whole system alters the extent of this cancellation and thus the relative magnitudes and phases of the two terms, but not their resultant. Individually the terms are not 'Galilean invariant'. The effect of each term in isolation can be misleading in attempting to diagnose the magnitude and even the sign of the vertical velocity.

A simple example of these difficulties is given by the most unstable Eady baroclinic instability mode, independent of the latitudinal direction, on a total wind shear ΔU. For the case when the surface wind, U_0, is zero, the phases and amplitudes of the terms $\mathscr{A}$ and $\mathscr{B}$ and their resultant $\mathscr{R}$ are given in Fig. 1. $\mathscr{A}$ and $\mathscr{B}$ are in phase at the surface, but their phase difference is more than 100° at the mid level and 150° at the top. The amplitudes show the terms $\mathscr{A}$ and $\mathscr{B}$ to be equal at the surface but elsewhere in the lower half $\mathscr{B}$ is dominant. A change in the surface wind produces different phases and amplitudes for $\mathscr{A}$ and $\mathscr{B}$, though not for $\mathscr{R}$. For example, taking $U_0 = -\Delta U/4$ gives $|\mathscr{A}| \simeq 2\frac{1}{2}|\mathscr{B}|$ at the surface and makes $\mathscr{A}$ dominant in the lower levels. $U_0 = +\Delta U/4$ gives $|\mathscr{B}| \simeq 2\frac{1}{2}|\mathscr{A}|$ at the surface.

The difficulty associated with the cancellation between $\mathscr{A}$ and $\mathscr{B}$ is made worse by the fact that $\mathscr{A}$ involves differentiation in the vertical and $\mathscr{B}$ is a horizontal Laplacian. In practical situations, confident qualitative determination of the resultant forcing of vertical velocity probably requires at least semi-quantitative calculations involving variables at more than one level of the atmosphere.

3. Comparison with Sutcliffe's theory

One may easily repeat the work of Sutcliffe using continuous representation in the vertical rather than two layers. Instead of Eq. (2) one would obtain $f^2(\partial^2 w/\partial z^2) = 2f(\partial\mathbf{V}_g/\partial z).\nabla\xi_g$. Assuming that $\partial^2 w/\partial z^2$ and w are of opposite sign, this form predicts upward motion whenever the thermal wind is directed towards decreasing vorticity.

For simplicity, Sutcliffe omitted adiabatic cooling and warming in the thermodynamic equation and so the term $N^2\nabla_h^2 w$ does not occur on the l.h.s. This is probably not serious for *qualitative* interpretation provided that the forcing is approximately sinusoidal with length scale greater than or equal to the Rossby radius of deformation L_R, $= NH/f$ where H is a vertical height scale.

Sutcliffe reduced the forcing to just one term, but at the expense of neglecting forcing terms which may be written

$$G = -2\frac{g}{\theta_0}\left[D_1\left(\frac{\partial^2\theta}{\partial x^2}-\frac{\partial^2\theta}{\partial y^2}\right)+2D_2\frac{\partial^2\theta}{\partial x\partial y}\right].$$

Here D_1, D_2 are the geostrophic deformation components:

$$D_1 = \partial u_g/\partial x = -\partial v_g/\partial y, \quad D_2 = \tfrac{1}{2}(\partial v_g/\partial x + \partial u_g/\partial y).$$

If the dilatation axis is at an angle α with a fixed axis, and the magnitude of the total deformation is D, it can be shown that $G = -8fD^2(\partial\alpha/\partial z)$. Thus the omitted term depends on the presence of deformation and on the rotation of the dilatation axis with height. As Sutcliffe pointed out, this term is certainly important in frontal regions. Indeed for the classical confluence frontogenesis model it is the sole forcing of vertical motion. If the direction of the thermal wind is almost independent of height then the advection of the thermal pattern may be determined from the surface flow and, even in regions of large deformation, the omitted term may be negligible.

We may note that the Sutcliffe form for the forcing is exact for the two-dimensional Eady wave discussed in the previous section. The resultant forcing is $\mathscr{R} = 2f(\Delta U/H)(\partial^2 v/\partial x^2)$. This is in agreement with Fig. 1 and, for a given shear ΔU, is clearly independent of the surface wind U_0.

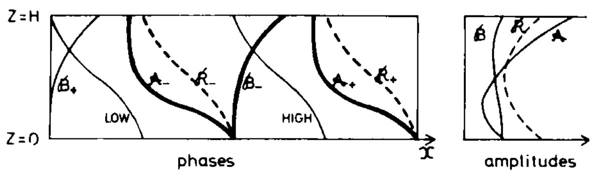

Figure 1. Phases and amplitudes of the forcing terms in the ω-equation for the most unstable Eady wave independent of y and for zero basic zonal flow at the ground. The phases are shown in an x, z cross-section with the positions of the pressure low and high at each level also indicated. The vorticity advection term, $\mathscr{A}$, the thermal advection term, $\mathscr{B}$ (see Eq. (2)), and the resultant of the two, $\mathscr{R}$, have their maxima and minima indicated by + and −. Positive forcing may be associated with downward motion and negative forcing with upward motion.

4. A REDERIVATION OF THE ω-EQUATION

In order to derive a form of the ω-equation that is easily applicable to synoptic cases, we shall first consider the tendency of geostrophic motion to destroy thermal wind balance. For consideration of the balance $f(\partial v_g/\partial z) = (g/\theta_0)(\partial\theta/\partial x)$, we use the y equation of motion and the potential temperature equation at the level of approximation of quasi-geostrophic theory:

$$(\partial/\partial t+\mathbf{V}_g.\nabla)v_g+fu_{ag} = 0 \quad \text{and} \quad (\partial/\partial t+\mathbf{V}_g.\nabla)\theta+w(d\Theta/dz) = 0 \quad . \quad (3)$$

If the ageostrophic motion (u_{ag}, w) is for the moment neglected, it is easily shown that

$$(\partial/\partial t+\mathbf{V}_g.\nabla)(g/\theta_0)(\partial\theta/\partial x) = -(\partial/\partial t+\mathbf{V}_g.\nabla)f(\partial v_g/\partial z) = Q_1,$$

where $Q_1 = -(g/\theta_0)(\partial\mathbf{V}_g/\partial x).\nabla\theta$.

Thus, in relation to a fluid particle moving with the geostrophic flow, we see that geostrophic motion destroys itself by changing the two parts of the thermal wind balance equally but in opposite directions.

In quasi-geostrophic theory, the role of ageostrophic motion is to restore thermal wind balance which the geostrophic motion is tending to destroy. If we now consider the ageostrophic terms also in Eq. (3), eliminating the time derivatives gives

$$N^2(\partial w/\partial x)-f^2(\partial u_{ag}/\partial z) = 2Q_1 \quad . \quad . \quad . \quad (4)$$

Geostrophic balance is restored by differential vertical motion tending to change the horizontal temperature gradient and differential ageostrophic horizontal velocity tending to change the vertical derivative of the horizontal wind. Thus for Q_1 positive one would usually expect geostrophic balance to be maintained by upward motion increasing with x, and by ageostrophic x velocity decreasing with height. It is possible that one of these processes could be of the opposite sign because of the three-dimensionality of the flow, but this would have to be compensated by larger values of the other process. Thus a simplified form of Eliassen's (1962) cross-frontal circulation theory is applicable even when the flow is three-dimensional.

One may describe the geostrophic forcing as direct if it tends to result in warm air rising, cold air descending and horizontal motion towards low pressure. Thus a forcing is direct if, following a fluid particle moving with the geostrophic velocity, the geostrophic motion tends to increase a pre-existing temperature gradient or, equivalently, if it tends to decrease the vertical shear in the horizontal geostrophic wind. It is indirect if the geostrophic tendencies following a fluid particle are of the opposite sign.

To obtain the full three-dimensional ageostrophic motion we must write the y equation equivalent to Eq. (4) and the continuity equation. The full set is:

$$\begin{aligned} N^2(\partial w/\partial x)-f(\partial u_{ag}/\partial z) &= 2Q_1 \\ N^2(\partial w/\partial y)-f(\partial v_{ag}/\partial z) &= 2Q_2 \quad . \quad . \quad . \quad (5) \\ \partial u_{ag}/\partial x+\partial v_{ag}/\partial y+\partial w/\partial z &= 0 \end{aligned}$$

where $$\mathbf{Q} = (Q_1, Q_2) = (-(g/\theta_0)(\partial \mathbf{V}_g/\partial x).\nabla\theta, -(g/\theta_0)(\partial \mathbf{V}_g/\partial y).\nabla\theta) \quad . \quad (6)$$

Eliminating the horizontal ageostrophic velocities gives the vertical velocity equation

$$N^2\nabla_h^2 w+f^2(\partial^2 w/\partial z^2) = 2\nabla.\mathbf{Q} \quad . \quad . \quad . \quad (7)$$

This may be shown to be exactly equivalent to the usual form of the ω-equation (Eq. (2)). The l.h.s. is identical, but the two terms on the r.h.s. have been combined in a rather suggestive manner.

In quasi-geostrophic theory, on an f-plane vertical velocity is forced solely by the divergence of **Q**.

Q is a constant times the vector rate of change of horizontal potential temperature gradient on a fluid particle implied by the geostrophic motion alone. For ease of understanding and application, to determine **Q** at a point we may take rectangular Cartesian axes with the x axis tangential to the potential temperature contour at that point and the y axis pointing towards colder air. Then $\partial\theta/\partial x = 0$, and Eq. (6) becomes

$$\mathbf{Q} = \{-(g/\theta_0)(\partial v_g/\partial x)(\partial\theta/\partial y), -(g/\theta_0)(\partial v_g/\partial y)(\partial\theta/\partial y)\} \quad . \quad (8)$$

The two simple motions illustrated in Fig. 2 show how in this coordinate system Q_1 is related to the horizontal shear and Q_2 to the diffluence or confluence of the geostrophic motion. For application to synoptic situations we note that Eq. (8) may be written

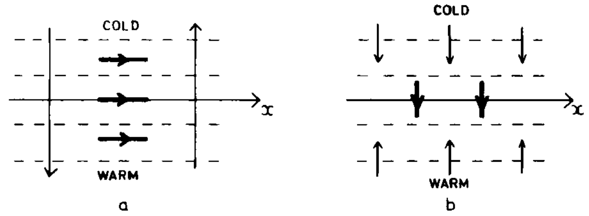

Figure 2. Illustration of the forcing terms when the potential temperature (dashed contours) is a function of y only with positive y pointing towards colder air. The motion is shown by light arrowed lines and **Q** by heavy arrowed lines. (a) Horizontal shear: v_g an increasing function of x only. Q_1 is positive and Q_2 zero (b) Confluence: v_g a decreasing function of y only. Q_1 is zero and Q_2 negative.

$$\mathbf{Q} \propto (1/L)\nabla v_g \qquad (9)$$

where L is the separation of potential or actual temperature contours.

For horizontal length scales large compared with the Rossby radius, L_R, the l.h.s. of Eq. (7) is dominated by the second term. This implies that the vertical shears in the wind are adjusted by the ageostrophic flow to match the changes in the thermal gradient implied by the geostrophic flow. For horizontal length scales comparable with L_R, both the wind and thermal fields are adjusted by the ageostrophic flow.

Before discussing application of these results to synoptic situations, for completeness we include the variation of the Coriolis parameter with latitude as modelled by the usual β-effect. This gives an extra forcing term in Eq. (7): $\beta f(\partial v_g/\partial z) = \beta(g/\theta_0)(\partial\theta/\partial x)$, and corresponds, in the northern hemisphere, to ascent in the northerly thermal wind westward of a thermal trough and descent in the southerly thermal wind eastward of it. Taking a vertical difference in the north–south wind of $20\,\mathrm{m\,s^{-1}}$ over a tropospheric depth $H \sim 10\,\mathrm{km}$ and approximating the l.h.s. of Eq. (7) by $\pi^2 f^2 w/H^2$ gives vertical motion of the order of $\frac{1}{3}\,\mathrm{cm\,s^{-1}}$ for mid-latitude values of f and β.

5. Application to the understanding of synoptic situations

As is customary for qualitative application, as a first approximation we may assume that regions where **Q** is divergent (i.e. $N^2\nabla_h^2 w + f^2(\partial^2 w/\partial z^2) > 0$) correspond to descent. This implies shrinking of columns below, and the creation of anticyclonic vorticity. Similarly below regions where **Q** is convergent cyclonic vorticity is created.

Returning to Fig. 2(a), **Q** is convergent for x positive and divergent for x negative. We thus expect the 'northward' moving air to ascend and the 'southward' moving air to descend. In Fig. 2(b), **Q** is convergent in the warmer air and divergent in the colder air. The warmer air is therefore predicted to ascend and the colder air to descend.

Standard meteorological situations provide a good illustration of the application of the theory described in this paper. In Fig. 3(a) are shown the height contours for the entrance and exit of a jet. Clearly, in the entrance region, cross-stream gradients in potential temperature on a fluid particle would be enhanced on the assumption of geostrophic motion. Thus the **Q** vectors are approximately as indicated. **Q** is divergent on the left (looking along the flow) and convergent on the right, implying downward motion and the creation of anticyclonic vorticity below on the left side and upward motion with the creation of cyclonic vorticity below on the right. In the exit region, the geostrophic motion tends to weaken the cross-stream temperature gradient which forces the creation of cyclonic vorticity below on the left, and anticyclonic on the right. In the special case that temperature contours are almost identical with height contours this case can be correctly diagnosed using the Sutcliffe development theory (Sutcliffe and Forsdyke 1950).

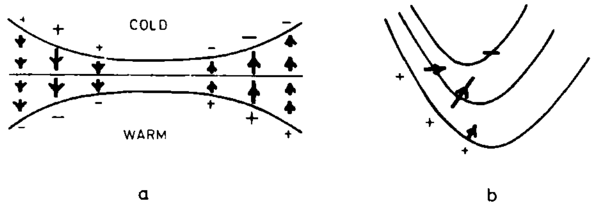

Figure 3. Two standard synoptic situations. (a) The height contours for a jet entrance and exit. (b) Temperature contours in a diffluent trough. Horizontal shears in the geostrophic velocity field are assumed to be dominated by the thermal wind contribution.
The arrowed lines indicate estimated **Q** vectors. Convergence and divergence of **Q** are given by − and +. They indicate the forcing of upward and downward motion respectively.

In Fig. 3(b) are shown typical tropospheric temperature (or 1000–500 mb thickness) contours for a diffluent trough. Assuming that low-level horizontal velocity shears are negligible, higher in the atmosphere velocity shears may be deduced from the thermal wind. The **Q** vectors must be approximately as shown. The strong convergence implies the forcing of cyclonic vorticity below in the cold air. The weaker divergence implies smaller generation of anticyclonic vorticity below in the warm air.

Finally we show a chart (Fig. 4) for an actual synoptic situation at 00 GMT on

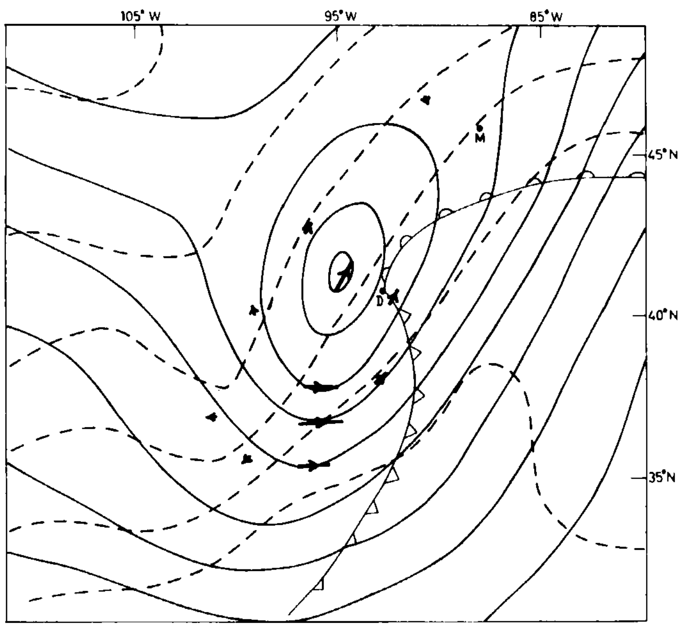

Figure 4. The 700 mb chart of height and temperature contours for the Great Plains region of N America at 00 GMT on 10 Nov. 1975. Height contours, every 3 decametres, are shown by continuous lines; temperature contours, every 4 deg C, are indicated by dashed lines. The surface pressure minimum was 994 mb, centred near Des Moines, Iowa (indicated by D). A surface frontal analysis is also marked. Estimates of the vector **Q** are given. 12 hours later the surface pressure minimum was 981 mb centred near Marquette, Michigan (M).

10 November 1975 when a major system was developing over the Mid-West of North America. The 700 mb level was chosen both because it should give a good indication of the forcing of vertical velocity in the lower half of the atmosphere, and because the crucial cross-isentropic component of velocity is not masked by large 'thermal' winds. The surface pressure minimum at this time was 994 mb. Its position, and a frontal analysis, are indicated

in Fig. 4. The closed vortex in the height field leads to **Q** vectors approximately along the isentropes. The temperature gradient in the cold-frontal region and the strong tendency to increase this gradient leads to large magnitude vectors oriented approximately as shown. Estimates of other smaller magnitude vectors are indicated. We thus expect a vigorous cold-frontal circulation with ascent at the surface front and strong descent behind, and general rising motion in the warm air ahead of the system. This is in agreement with pressure rises larger than 6 mb in 3 hours behind the cold front, severe storms including tornadoes at the cold front, and in 12 hours the surface pressure low deepening to 981 mb and moving to the position indicated. The region of the analysed warm front does not contain large forcing of vertical motion but appears to be connected more with the saturated ascent ahead of the system.

6. Final comments

The work described here is not a new theory, but merely an attempt to draw together the theories on the forcing of ageostrophic motion and vertical velocity which are used in dynamical and synoptic meterology. By writing the forcing term in a new way, it has been possible to infer the synoptic distribution of vertical velocity by consideration of a height and temperature field chart at a constant pressure level. The intention has not been to discount the usefulness of the Sutcliffe development theory, but rather to examine the error involved in the theory and to describe how more accuracy may be obtained without much more complexity.

We have applied quasi-geostrophic theory even in regions of large vorticity where it is not strictly applicable. In a subsequent paper (Hoskins and Draghici 1977) it will be shown that, in such regions, only slight modification of the ageostrophic velocity equations (Eqs. (5)–(6)) is required.

Although weather forecasting is now dominated by the products of numerical models it is hoped that the work described here is of practical relevance. For initialization procedures it is essential to know the ingredients that produce the vertical velocity field and development. Such theories are also useful for evaluating the numerical product, particularly in its underestimation of 'the weather' on the frontal scale.

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